Ising model on hyperbolic lattice studied by corner transfer matrix renormalization group method

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Abstract. We study two-dimensional ferromagnetic Ising model on a series of regular lattices, which are represented as a tessellation of polygons with $p \geq 5$ sides, such as pentagons ($p = 5$), hexagons ($p = 6$), etc. Such lattices are on hyperbolic planes, which have constant negative scalar curvatures. We calculate critical temperatures and scaling exponents by use of the corner transfer matrix renormalization group method. As a result, the mean-field like phase transition is observed for all the cases $p \geq 5$. Convergence of the calculated transition temperatures with respect to p is investigated towards the limit $p \to \infty$, where the system coincides with the Ising model on the Bethe lattice.

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1. Introduction

The Ising model has been extensively investigated because of its simplicity in definition and wide applicability to real magnetic materials. The model is exactly solvable in two dimensions (2D) under appropriate conditions [1, 2]. For the study of insolvable cases, such as the cross-bond Ising model and three-dimensional (3D) models, a variety of numerical methods have been developed, such as Monte Carlo simulations [3], Lanczos diagonalization of row-to-row transfer matrices, and Baxter's method of corner transfer matrices (CTMs) [2]. One of the recent technical progress in numerical study is establishment of the density matrix renormalization group (DMRG) method [4, 5, 6]. The method is applicable to 2D classical lattice models including the Ising model [7] and is of use for the study of higher-dimensional lattice models [8, 9, 10, 11, 12, 13].

It is widely believed that the phase transition of the Ising model belongs to the so-called Ising universality class provided that the system is uniform and on planar 2D lattices. This universality can be violated if the lattice is in curved spaces, where typical

examples are the lattices represented as regular tessellation of polygons in the hyperbolic plane, which has a constant negative scalar curvature [14, 15, 16]. As was pointed by Chris Wu *et al.*, boundary effects are non-negligible below the transition temperature on such hyperbolic lattices even in the thermodynamic limit [17, 18]. d'Auriac *et al.* investigated the bulk and boundary states and discussed their difference [19]. A recent Monte Carlo (MC) study by Shima and Sakaniwa for the Ising model on one of the hyperbolic lattices shows that the critical behavior in the ferromagnetic-paramagnetic transition deep inside the system is mean-field like [20, 21]. Their result is in accordance with the bulk property discussed by d'Auriac *et al.* [19].

The size of the system treated by the MC simulations on the hyperbolic lattices is limited by an exponential grow of the number of lattice points. Some sort of renormalization group scheme is required under such a situation. Quite recently we have applied the corner transfer matrix renormalization group (CTMRG) method [22, 23] to a particular hyperbolic lattice which consists of pentagons ($p = 5$) [24]. The CTMRG method enables precise estimation of the bond energy and the magnetization at the center of a sufficiently large system. Ferromagnetic boundary conditions is assumed to observe the bulk property. As a result, we have confirmed the mean-field like behavior of the phase transition for the studied case $p = 5$. In this article we extend our previous study by considering hyperbolic lattices that consist of arbitrary "p-gons" with $p > 5$, such as hexagons ($p = 6$), heptagons ($p = 7$), etc. For the study of large p cases, we introduce a novel partial sum technique to the CTMRG method.

We calculate transition temperature $T_{\rm c}$ for each case $p \geq 5$ as well as related critical exponents α, β and δ, respectively, associated with the specific heat, the spontaneous and induced magnetization. We then observe convergence of $T_{\rm c}$ toward the limit $p \to \infty$, where the system corresponds to the Ising model on the Bethe lattice. In the next section we explain detail of the model on the hyperbolic lattices. We observe the structure of the lattices from the view point of the corner transfer matrix formalism. Numerical results are presented in Sec. 3, where we calculate the critical temperatures and the critical exponents. The conclusions are summarized in the last section.

2. Structure of the system on hyperbolic lattice

Consider a series of infinite-size lattices that consist of regular polygons with $p \geq 5$ sides, which are called as 'p-gons'. Each lattice is represented as a tessellation of the p-gons on an infinite plane with a constant negative scalar curvature. One can classify this type of lattices by a pair of integers (p, q), where the coordination number q represents the number of the neighboring lattice points. In the following we consider the ($p \geq 5$, $q = 4$) lattices, including the pentagonal lattice $(5, 4)$, the hexagonal one $(6, 4)$, the heptagonal one $(7, 4)$, etc. We also treat a square lattice $(4, 4)$ defined on the flat plane for comparison.

As an example, we draw the pentagonal lattice $(5, 4)$ in the left part of Fig. 1, where the infinite area of the hyperbolic plane is mapped into the Poincaré disc. All

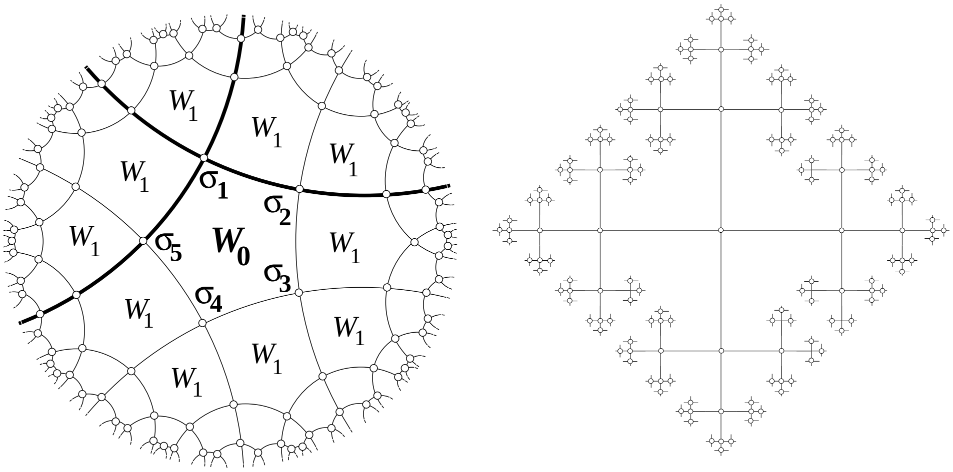

Figure 1. Left: the Ising model on the pentagonal lattice $(5,4)$ which is drawn in the Poincaré disc. The open circles represent the Ising spins σ_i. Note that each pentagon has the same size and shape. Right: the Bethe lattice of the coordination number $q=4$ is equivalent to the $(\infty,4)$ lattice.

arcs in the figure represent geodesics that are perpendicular to the bounding circle. Two geodesics drawn by the thick arcs cross one another at a lattice point. Note that by these two geodesics, the whole system is divided into four equivalent semi-infinite parts, which are called as the *quadrants* or *corners*. As another typical example, we draw the $(\infty,4)$ lattice in the right part of Fig. 1. This lattice is merely the Bethe lattice with the coordination number $p=4$. Note that the Hausdorff dimension of these ($p\geq 5$, 4) lattices is infinite.

Consider the Ising model on the ($p\geq 5$, 4) lattice, where on each lattice point there is an Ising spin $\sigma_i = \uparrow\downarrow$. If only the neighboring Ising interactions are assumed, the Hamiltonian of the system is represented as

$$\mathcal{H} = -J\sum_{\{i,j\}}\sigma_i\,\sigma_j - H\sum_{\{i\}}\sigma_i \;, \tag{1}$$

where the summation $\{i,j\}$ runs over all nearest-neighbor spin pairs. We assume that the interaction is ferromagnetic ($J>0$). The external magnetic field H acts on each spin site uniformly. For latter conveniences of expressing the partition function, let us introduce the weight $w(\sigma_i\,\sigma_j)$ assigned to the neighboring spin pair $\{i,j\}$

$$w(\sigma_i\,\sigma_j) = \exp\left[\beta J\frac{\sigma_i\,\sigma_j}{2} + \beta H\frac{\sigma_i+\sigma_j}{8}\right] \tag{2}$$

with $\beta = 1/k_{\rm B}T$. The Boltzmann weight of the whole system is then expressed as

$$\exp(-\beta\mathcal{H}) = \prod_{\{i,j\}}\left[w(\sigma_i\,\sigma_j)\right]^2 \;. \tag{3}$$

Since each bond is shared by two p-gons, it is possible to assign a local Boltzmann weight for each p-gon. Let us focus on the p-gon, where spins on its edges are labeled by

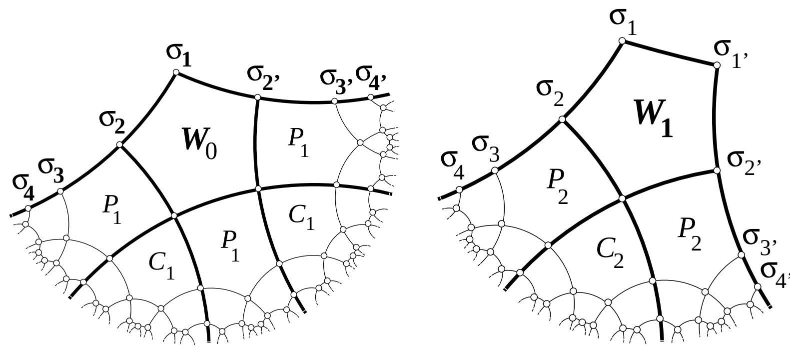

Figure 2. A corner transfer matrix $C(\ldots\sigma_3\sigma_2\sigma_1|\sigma_{1'}\sigma_{2'}\sigma_{3'}\ldots)$ of the case $p = 5$ shown on the left side consists of a face weight W_0, two CTMs of smaller size C_1, and three half-row transfer matrices P_1. Each HRTM $P(\ldots\sigma_3\sigma_2\sigma_1|\sigma_{1'}\sigma_{2'}\sigma_{3'}\ldots)$ shown in the right has an analogous substructure.

σ_1, σ_2, ..., and σ_p, as shown in the left side of Fig. 1, where the case $p = 5$ is drawn as an example. The Boltzmann weight assigned to the p-gon, which is called as the 'face weight', is then expressed as

$$W(\sigma_1\,\sigma_2\,\sigma_3\,\ldots\,\sigma_p) = w(\sigma_1\,\sigma_2)\,w(\sigma_2\,\sigma_3)\ldots w(\sigma_{p-1}\,\sigma_p)\,w(\sigma_p\,\sigma_1)\,. \tag{4}$$

It is straightforward that one can assign the same weight W for all the p-gons in the system. We have thus represented the Ising model on the $(p\geq 5,\, 4)$ lattices as a special case of the interaction-round-a-face (IRF) model, which regards the 'face' as the unit of the system [2].

The partition function of a finite-size system is represented as

$$\mathcal{Z} = \sum_{\{\sigma\}} \prod W\,, \tag{5}$$

where the sum is taken over all configurations of the spins. The product runs over all the face weights contained in the system starting from a weight, which is shown as W_0 on the left in Fig. 1, at the center of the system. Around W_0 there are $2p$ number of neighboring weights W_1 in the first shell, $4p(p-3)$ number of W_2 in the second shell, etc. The number of the weights and sites in the α-th shell increases exponentially with α.

For the calculation of the partition function $\mathcal{Z}$, we introduce the corner transfer matrix (CTM) denoted by C that represents the Boltzmann weight for each quadrant of the system [2]. By use of the CTM, the partition function is expressed as the trace

$$\mathcal{Z} = \mathrm{Tr}\, C^4 \tag{6}$$

of the density matrix $\rho = C^4$. In the following we use the common notations in the CTMRG method [22, 23, 24]; see the detail in Ref. [24].

Let us consider a finite-size system that contains the lattice points up to the N-th shell, where the ferromagnetic boundary condition is imposed at the lattice border. The left side of Fig. 2 shows the structure of the CTM of the system for the case $p = 5$. The CTM C contains a face weight labeled by W_0, two CTMs of the smaller size labeled by

C_1, and three parts labeled by P_1 that corresponds to the so-called half-raw transfer matrix (HRTM). The right side of Fig. 2 shows similar substructure of the HRTM for $p = 5$. Looking at these figures, one finds a recursive relation between the CTMs and the HRTMs. If one has C and P of a certain linear size, one can obtain the extended ones C' and P' by the following fusion process [24]

$$\begin{aligned} C' &= W \cdot P \cdot (C \cdot P)^{p-3} \\ P' &= W \cdot P \cdot (C \cdot P)^{p-4} \ , \end{aligned} \tag{7}$$

which increases the linear size of C and P by one. Note that if ferromagnetic boundary condition is imposed for both C and P, the extended ones C' and P' are also subject to the same boundary condition. Repeating this fusion process, one can obtain CTMs and HRTMs of arbitrary linear sizes provided that these matrices can be stored to a computational machine. This storage limitation can be removed by use of the renormalization group (RG) transformation in the density matrix scheme [4, 5, 6]. As a result, the matrices C and P are *renormalized* into effective ones $\tilde{C}$ and $\tilde{P}$, whose matrix dimension is at most $2m$ where m is the number of states kept for each block spin [4].

One-point functions at the center of the system are easily calculated by use of $\tilde{C}$ thus obtained by way of sufficient number of iterative extensions and the RG transformations. For example, the spontaneous magnetization is calculated as

$$\mathcal{M} = \langle \sigma \rangle = \frac{\mathrm{Tr}\ \sigma\, \tilde{C}^4}{\mathrm{Tr}\ \tilde{C}^4} \ , \tag{8}$$

where σ denotes the Ising spin at the center of the system. For the bond energy, we similarly express it as

$$\mathcal{U} = -J \langle \sigma\tau \rangle = -J\, \frac{\mathrm{Tr}\ \sigma\tau\, \tilde{C}^4}{\mathrm{Tr}\ \tilde{C}^4} \ , \tag{9}$$

where τ is a neighboring spin to σ. From the calculated $\mathcal{U}$, the specific heat can be obtained by taking the numerical differential $\mathcal{C} = \partial \mathcal{U} / \partial T$.

3. Numerical Results

Numerical analysis is carried out for the cases $p \geq 5$. Because of the product structure of the local weight W shown in Eq. 4, the fusion process expressed by Eq. 7 can be performed for arbitrary large p without any increase of computational memory. We keep at most $m = 50$ states for the block spin variable during the CTMRG calculations. For all the cases investigated here, the density matrix eigenvalues decay very fast even at the transition temperature. This is in contrast to the relatively slow decay observed in the square lattice models [28]. Thus actually $m = 10$ is sufficient for the calculation of the magnetization $\mathcal{M}$ as well as the bond energy $\mathcal{U}$.

The left side of Fig. 3 shows the temperature dependence of the spontaneous magnetization. We have chosen dimensionless parameters $k_{\mathrm{B}} = J = 1$. For comparison,

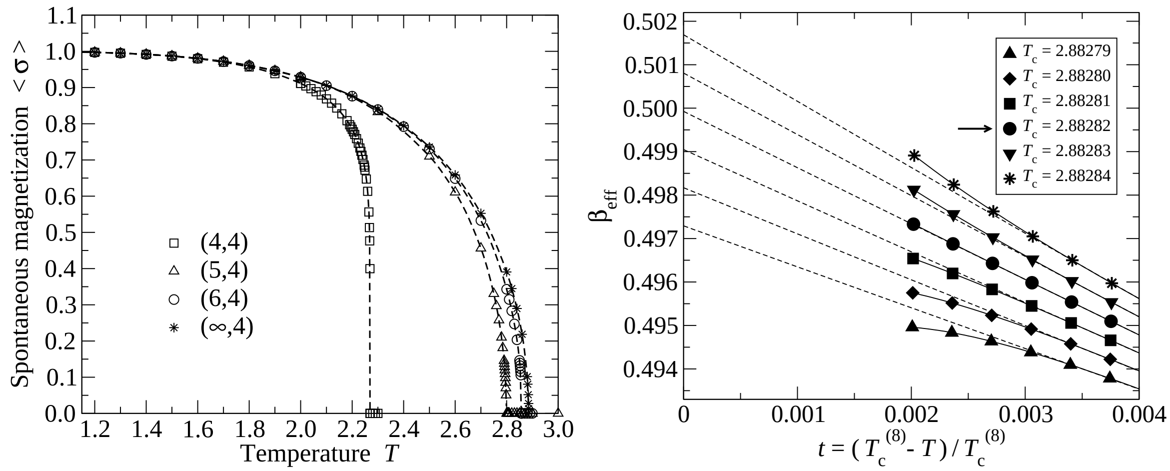

Figure 3. Left: the spontaneous magnetization $\mathcal{M}$ with respect to temperature T at $H = 0$. Right: the t-dependence of the effective critical exponent in Eq. (10) for the case of $(8, 4)$ lattice.

Table 1. The calculated critical temperatures $T_{\rm c}^{(p)}$.

(p, q)	$(4, 4)$	$(5, 4)$	$(6, 4)$	$(7, 4)$	$(8, 4)$	$(9, 4)$
$T_{\rm c}^{(p)}$	$2/\ln\left(\sqrt{2}+1\right)$	2.79908	2.86050	2.87754	2.88282	2.88457
(p, q)	$(10, 4)$	$(11, 4)$	$(12, 4)$	$(15, 4)$	$(30, 4)$	$(\infty, 4)$
$T_{\rm c}^{(p)}$	2.88519	2.88533	2.88538	2.88539	2.88539	$1/\ln\sqrt{2}$

we also draw $\mathcal{M}$ for the case of the Bethe lattice with the coordination number $q = 4$. In the critical region below the transition temperature $T_{\rm c}^{(p)}$, the magnetization behaves as $\mathcal{M} = f(t)\, t^{\beta}$, where $f(t)$ is a slowly varying function of $t = (T_{\rm c}^{(p)} - T)/T_{\rm c}^{(p)}$, the rescaled temperature deviation from $T_{\rm c}^{(p)}$. In order to estimate $T_{\rm c}^{(p)}$ precisely, we plot the effective critical exponent

$$\beta_{\rm eff}(t) = \frac{\partial}{\partial \ln t} \ln \mathcal{M} = \beta + \frac{\partial}{\partial \ln t} \ln\, f(e^{\ln t}) = \beta + \frac{f'}{f}\, t + \ldots \tag{10}$$

in a very small t region. The right side of Fig. 3 shows the effective exponent $\beta_{\rm eff}(t)$ thus calculated for the case $p = 8$. From the trial critical temperatures listed in the inset, $T_{\rm c}^{(8)} = 2.88282$ gives the best linear fit. We have applied the same procedure for all p that we have chosen. The results are listed in Table 1, where $\beta_{\rm eff}(0) = \beta \cong \frac{1}{2}$ is confirmed for all the cases. Figure 4 shows the t-dependence of $\mathcal{M}^2$ (left) and $\mathcal{M}^8$ (right). It is obvious that the mean-field exponent $\beta = \frac{1}{2}$ is observed for all the cases $p \geq 5$, whereas the Ising universality class $\beta = \frac{1}{8}$ is realized for the square lattice $(4, 4)$ only.

At the calculated $T_{\rm c}^{(p)}$, let us observe the induced magnetization $\mathcal{M}$ with respect to the applied field H. From the scaling relation $\mathcal{M} \propto H^{1/\delta}$, another critical exponent δ

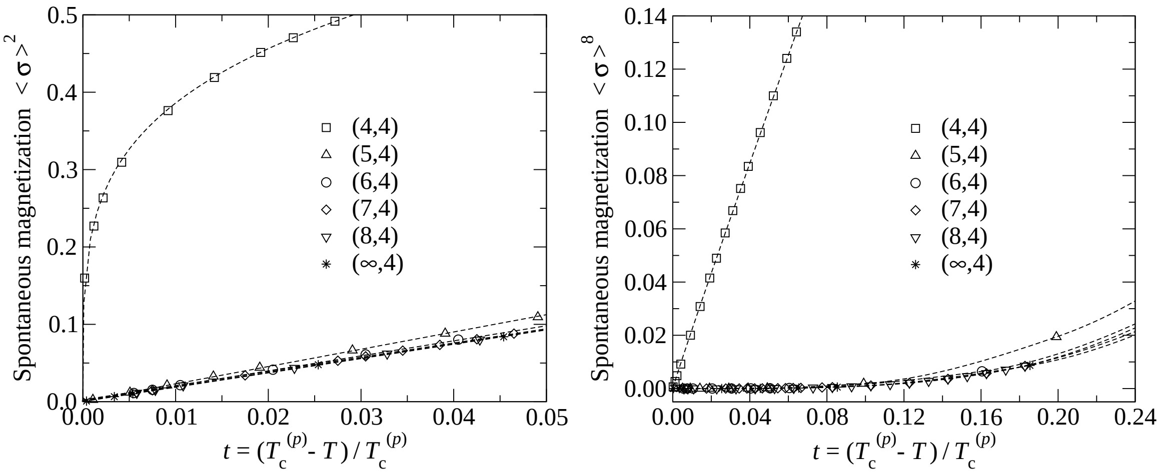

Figure 4. The t-dependence of $\mathcal{M}^2$ (left) and $\mathcal{M}^8$ (right). The mean-field exponent $\beta = \frac{1}{2}$ is observed for $p \geq 5$, whereas $\beta = \frac{1}{8}$ exclusively for $p = 4$.

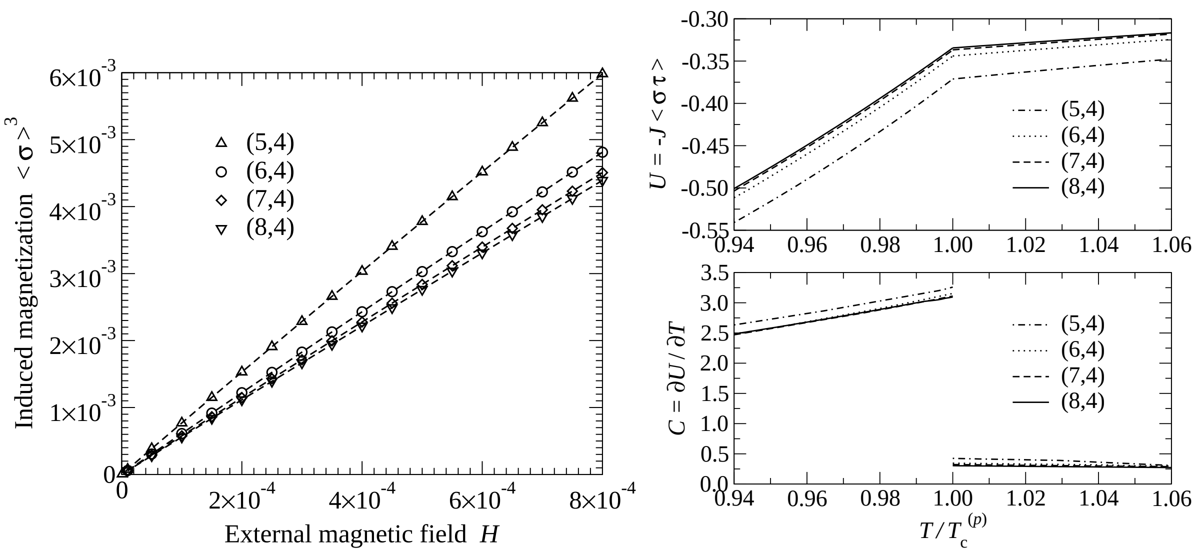

Figure 5. Left: Induced magnetization at $T_c^{(p)}$ with respect to the applied magnetic field H. Right: the upper and lower panels, respectively, display singularity of the internal energy $\mathcal{U}$ and the specific heat $\mathcal{C}$ around in the critical region.

can be extracted. The left side of Fig. 5 shows the linearity of $\mathcal{M}^3$ with respect to small external magnetic fields H calculated at the critical temperature $T_c^{(p)}$ listed in Table 1. It is apparent that δ is equal to 3, which supports the mean-field like behavior of the Ising model on the $(p, 4)$ lattices when $p \geq 5$.

To confirm the mean-field nature of the phase transition, we calculate the internal energy $\mathcal{U}$ by way of Eq. (9). The right side of Fig. 5 shows $\mathcal{U}$ with respect to the rescaled temperature $T/T_c^{(p)}$. For each case there is a cusp at $T = T_c^{(p)}$, and a linear dependence of $\mathcal{U}$ in the vicinity of $T_c^{(p)}$ supports the critical exponent $\alpha = 0$. There is a jump in specific heat.

Let us observe the convergence of $T_c^{(p)}$ with respect to p towards $T_c^{(\infty)} = 1/\ln\sqrt{2} =$

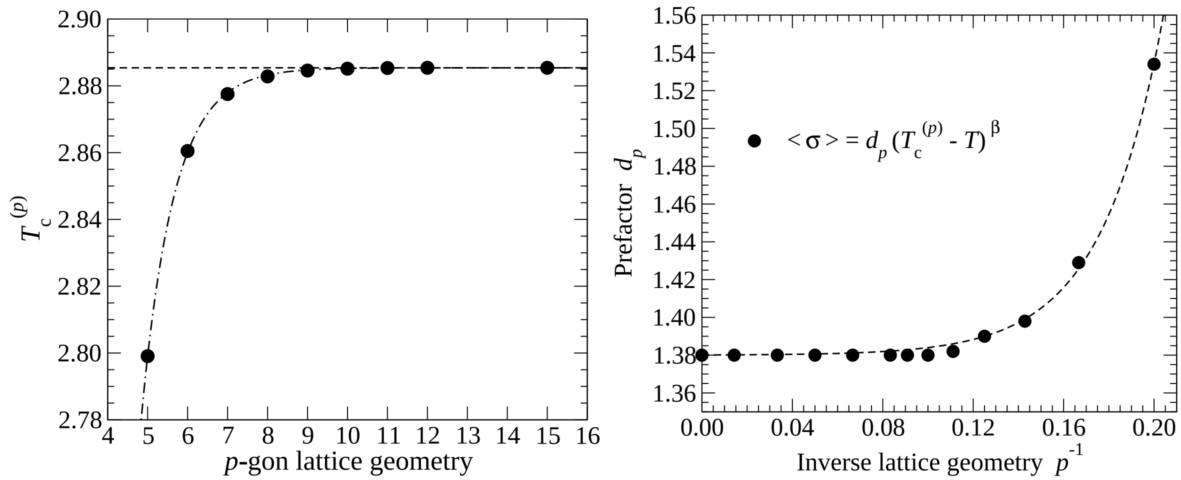

Figure 6. Left: the exponential dependence of $T_c^{(p)}$ with respect to p. The dashed horizontal line corresponds to the exact result on the Bethe lattice. Right: a scaling law of the prefactor d_p associated with temperature dependence of the spontaneous magnetization with respect to p.

2.88539. As shown on the left side of Fig. 6, the convergence is exponential

$$T_c^{(p)} - T_c^{(\infty)} \propto e^{-ap} \tag{11}$$

with respect to p. Fitting the plotted data for $5 \leq p \leq 8$, we have obtained the decay factor $a = 1.2543$. The prefactor d_p in the scaling relations

$$\mathcal{M} = d_p \, (T_c^{(p)} - T)^\beta \tag{12}$$

also shows a monotonous convergence to d_∞ as shown on the right side of Fig. 6. We have not obtained any appropriate fitting function of the p-dependence yet (the dashed line corresponds to an exponential fit).

4. Conclusions

We have calculated the magnetization, the internal energy and the specific heat of the Ising model on a series of ($p \geq 5$, 4) lattices on the hyperbolic planes. These quantities are observed at the center of the system with ferromagnetic boundary condition. We calculated the critical exponents and obtained $\alpha = 0$, $\beta = \frac{1}{2}$, and $\delta = 3$ for all the cases. Our result supports and complements previous predictions given by d'Auriac *et al.* [19], and independently by Shima *et al.* [20, 21]. The obtained results are in accordance with the fact that the Hausdorff dimension is infinite on the hyperbolic lattices and also with common knowledge that the mean-field like phase transition is observed above the critical dimension $d_c = 4$ [25].

The transition temperature $T_c^{(p)}$ of the Ising model on the $(p, 4)$ lattice converges exponentially fast towards $T_c^{(\infty)}$ with respect to increasing p. We have not yet clarified physical interpretation of this convergence. A renormalization group scheme given by Hilhorst *et al.* may provide some information to this question [29]. A recent numerical

renormalization group scheme suggested by Levin and Nave might be of use to find out an appropriate fixed-point Hamiltonian [30].

Recent study of the planar rotator (i.e. the classical XY) model on a hyperbolic lattice suggests that the mean-field like phase transition is not always realized for systems with the hyperbolic geometry [26]. Such XY model can be investigated by the generalized CTMRG method explained in this article [24] if appropriate boundary conditions are chosen [27].

Acknowledgments

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